Bose Glass in Large N Commensurate Dirty Boson Model

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The large N commensurate dirty boson model, in both the weakly and strongly commensurate cases, is considered via a perturbative renormalization group treatment. In the weakly commensurate case, there exists a fixed line under RG flow, with varying amounts of disorder along the line. Including $1/N$ corrections causes the system to flow to strong disorder, indicating that the model does not have a phase transition perturbatively connected to the Mott Insulator-Superfluid (MI-SF) transition. I discuss the qualitative effects of instantons on the low energy density of excitations. In the strongly commensurate case, a fixed point found previously is considered and results are obtained for higher moments of the correlation functions. To lowest order, correlation functions have a log-normal distribution. Finally, I prove two interesting theorems for large N vector models with disorder, relevant to the problem of replica symmetry breaking and frustration in such systems.

I. INTRODUCTION

The dirty boson problem, a problem of repulsively interacting bosons in a random potential, has been the subject of much theoretical work [1]. In the zero temperature quantum problem, the system can undergo a phase transition between a Bose glass phase and a superfluid phase. An action that may be used to describe this transition is

$$\int d^d x\, dt \left(+\partial_x \overline{\phi}(x,t)\partial_x \phi(x,t) + \partial_t \overline{\phi}(x,t)\partial_t \phi(x,t) + w(x)\overline{\phi}(x,t)\partial_t \phi(x,t) - U(x)\overline{\phi}\phi + g(\overline{\phi}\phi)^2 \right) \qquad (1)$$

This action describes a system with a number of phases. In the pure case, in which $U(x)$ is a constant, and $w(x)$ vanishes everywhere (the commensurate case), there is a phase transition from a gapped Mott insulator phase to a superfluid phase as U increases. If we consider a case in which $U(x)$ is not constant, but $w(x)$ still vanishes everywhere, there will be a sequence of transitions from a gapped Mott insulator, to a gapless (but with exponentially vanishing low energy density of states) Griffiths phase, and then to a superfluid phase. The Griffiths phase occurs due to the possibility of large, rare regions in which fluctuations in $U(x)$ make the system locally appear superfluid.

If we consider the case in which both $U(x)$ and $w(x)$ are fluctuating, there will also appear a Bose glass phase in which there is a density of states tending to a constant at low energy and an infinite superfluid susceptibility [2]. The physical basis for this phase is the existence of localized states, in which a competition between chemical potential and repulsion causes the system to desire a certain number of

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particles to occupy each localized state. There exist excitations involving adding or removing one particle from these states, and these excitations lead to the diverging susceptibility. However, it is clear that the Bose glass phase is very similar to the Griffiths phase, in that both involve regions of finite size. In the Griffiths phase one needs regions of arbitrarily large size, while in the Bose glass phase one only needs regions large enough to support a localized state, with a nonvanishing number of particles occupying that state, in order to produce the diverging susceptibility characteristic of the phase. In a system in which the disorder is irrelevant at the pure fixed point, so that the fluctuations in $U(x)$ and $w(x)$ scale to zero, one will still find a Bose glass phase as there will, with low probability, exist regions that can give rise to these localized states. Thus, the interesting question to answer is not whether the Bose glass phase exists, but whether there exists a fixed point at which fluctuations in $w(x)$ are weak, so that the critical exponents are near those of the pure MI-SF transition. The most likely alternative would be governed by the scaling theory of Fisher et. al., which has very different critical exponents [1,3]. We will refer to this scaling theory as the phase-only transition, as one assumes fluctuations in the amplitude of the order parameter are irrelevant at the critical point.

Recently, a large N generalization of equation (1) was considered in the restricted case $w(x) = 0$ [4]. We will refer to this case as the strongly commensurate case, while the situation in which $w(x)$ vanishes on average, but has nonvanishing fluctuations will be known as the weakly commensurate case. We consider a system defined by the partition function

$$\int \left(\prod_{x,t} \delta(\overline{\phi}_i(x,t)\phi^i(x,t) - N\sigma^2(x)) \right) [d\phi_i] e^{-S} \tag{2}$$

where

$$S = \int d^d x\, dt (\partial_x \overline{\phi}_i(x,t)\partial_x \phi^i(x,t) + \partial_t \overline{\phi}_i(x,t)\partial_t \phi^i(x,t) + w(x)\overline{\phi}_i(x,t)\partial_t \phi^i(x,t)) \tag{3}$$

Here, we have, for technical simplicity later, replaced the quartic interaction by a δ-function. For most of the paper, a δ-function interaction will be used. However, for generality in the last section, we will return to quartic interactions. The disorder in $U(x)$ will be replaced, in the δ-function case, by weak fluctuations in σ^2. We will consider $\sigma^2 = \sigma_0^2 + \delta\sigma^2$ where σ_0^2 is a constant piece used to drive the system through the phase transition and $\delta\sigma^2$ is a fluctuating piece.

The advantage of the large N formulation of the problem is that for any fixed realization of the problem one may exactly solve by system by finding the solution of the self-consistency equation

$$\sigma^2(x) = \langle x, t = 0 | (-\partial_x^2 - \partial_t^2 + w(x)\partial_t + \lambda(x))^{-1} | x, t = 0 \rangle \tag{4}$$

or

$$\sigma^2(x) = \int d\omega\, \langle x, t = 0 | (-\partial_x^2 + \omega^2 + iw(x)\omega + \lambda(x))^{-1} | x, t = 0 \rangle \tag{5}$$

where $\lambda(x)$ is a Lagrange multiplier field for enforce the δ-function constraint on the length of the spins. After solving the self-consistency equation, any correlation function can be found simply by finding the Green's function of a non-interacting field ϕ with action $\int d^d x\, dt\, \overline{\phi}(-\partial_x^2 - \partial_t^2 + w(x)\partial_t + \lambda(x))\phi$.

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In equation (4), we assume that the Green's function on the right-hand side has been renormalized by subtracting a divergent quantity. Specifically, we will take a Pauli-Villars regularization for the Green's function, and take the regulator mass to be very large, while adding an appropriate divergent constant to σ^2 on the left-hand side. The cutoff for the regulator is completely different from the cutoff for fluctuations in $\delta\sigma^2$ that will be used for the RG later; the cutoff for the regulator will be much larger than the cutoff for fluctuations in $\delta\sigma^2$ and $w(x)$, and will be unchanged under the RG.

In the previous work, exact results were obtained for the critical exponents for average quantities. In the present paper, I will first consider the problem in which $w(x) \neq 0$, although $w(x)$ vanishes on average (the weakly commensurate dirty boson problem). I will consider general problems of the large N system with terms linear in the time derivative. Next, a lowest order perturbative RG treatment will be used to consider critical behavior. Instanton corrections to the perturbative treatment will be briefly discussed after that. Returning to the strongly commensurate case, previous results on the fixed point will be extended to give results for higher moments of the correlation functions. Finally, as a technical aside, we consider the large N self-consistency equation in frustrated systems, and demonstrate that the self-consistency equation always has a unique solution, as well as considering the number of spin components needed to form a classical ground state in frustrated systems.

II. BOSE GLASS IN THE LARGE N LIMIT

Consider the following simple $0+1$ dimensional problem, at zero temperature ($\beta \to \infty$):

$$\int [d\phi_i(t)] \prod_t \delta(\overline{\phi}_i(t)\phi^i(t) - N\sigma^2) e^{\int_{-\beta/2}^{\beta/2}\{-\partial_t\overline{\phi}_i\partial_t\phi^i + A\overline{\phi}_i\partial_t\phi^i\}dt} \tag{6}$$

The solution of this problem in the large N limit via a self-consistency equation requires finding a λ such that

$$\sigma^2 = \int \frac{d\omega}{2\pi} \frac{1}{iA\omega + \omega^2 + \lambda} \tag{7}$$

Then, by contour integration, for $\lambda > 0$, we find $\sigma^2 = \frac{1}{\sqrt{A^2+4\lambda}}$. Then,

$$\lambda = \frac{1}{4}(\frac{1}{\sigma^4} - A^2) \tag{8}$$

Unfortunately, this result for λ leads to λ becoming negative for sufficiently large A. Although perturbation theory will not see a problem, this is the signal for the Bose glass phase. One must separate the self-consistency equation into two parts, one containing an integral over non-zero ω, and one containing the term for zero ω. One finds (for finite β)

$$\sigma^2 = \int \frac{d\omega}{2\pi} \frac{1}{iA\omega + \omega^2 + \lambda} + \frac{1}{\beta}\frac{1}{\lambda} \tag{9}$$

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Then, the self-consistency equation can always be solved using positive λ, but in the zero-temperature limit of the problem one will find that one needs λ to be of order $\frac{1}{\beta}$, and at zero-temperature, there will appear a zero energy state.

Considering the original statistical mechanics problem of equations (1,2), one will expect to see some non-zero density of these zero energy states, indicating the presence of a gapless Bose glass phase, with diverging superfluid susceptibility. Even if the fluctuations in $w(x)$ vanish at the fixed point, when the critical exponents are unchanged from the strongly commensurate problem, such zero energy states will exist as Griffiths effects leading to the appearance of a Bose glass phase near the superfluid transition.

To perform a renormalization group treatment of the model, we will first proceed in a perturbative fashion in the next section, ignoring such zero energy effects. For small fluctuations in $w(x)$, these zero energy states will be exponentially suppressed as will be discussed in the section after that.

III. PERTURBATIVE RG

We will follow the RG techniques used in previous work on the large N problem [4]. We will work near $2+1$ dimensions, specifically we will have $d = 2 + \epsilon$ space dimensions and 1 time dimension. We will at first work to one-loop in perturbation theory, which will, in some cases, correspond to lowest order in ϵ. Some results will be extended to all orders. A fixed line is found for the large N system. The fixed points are destroyed by $1/N$ corrections.

The RG is defined as follows: start with a system containing fluctuations in $\delta\sigma^2$ and $w(x)$ up to some wavevector Λ. Remove the high wavevector fluctuations in these $\delta\sigma^2$ and $w(x)$ to obtain a new system, with renormalized gradient terms ∂_x^2 and ∂_t^2 in the action, as well as renormalized low wavevector $\delta\sigma^2$ and $w(x)$ terms. Do this procedure so as to preserve the average low momentum Green's function, as well as the low wavevector fluctuations in $\lambda(x)$. See previous work [4] for more details.

If we are working in $2 + \epsilon$ space dimensions, and 1 time dimension, we can easily work out the naive scaling dimensions of the disorder. One finds that if we assume Gaussian fluctuations in the disorder, with

$$\langle \delta\sigma^2(p)\delta\sigma^2(q) \rangle = (2\pi)^2 \delta(p+q)S \tag{10}$$

$$\langle w(p)w(q) \rangle = (2\pi)^2 \delta(p+q)W \tag{11}$$

then S scales as length to the power $d - 2 = \epsilon$, while W scales as length to the power $2 - d = -\epsilon$. So, for $d > 2$ we find that disorder in σ^2 is relevant at the pure fixed point, while disorder in $w(x)$ is irrelevant. For $d < 2$ this is reversed.

Previously, a lowest order in ϵ calculation [4] considering only disorder in $\delta\sigma^2$ gave the following results. For a given problem, with fluctuations in $\delta\sigma^2$ up to wavevector Λ, and self-consistency equation

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$$\sigma^2(x) = \int d\omega \, \langle x, t = 0 | \left(-\partial_x^2 + \omega^2 + \lambda(x) \right)^{-1}) | x, t = 0 \rangle \tag{12}$$

one could define another problem, with fluctuations in $\delta\sigma^2$ only up to $\Lambda - \delta\Lambda$, with self-consistency equation

$$(1 - \frac{\delta\Lambda}{\Lambda} c_3 L)\sigma^2(x) = \int d^{(D-d)}\omega \, \langle x, t = 0 | \left(-(1 + \frac{\delta\Lambda}{\Lambda} c_2 L)\partial_x^2 + (1 + \frac{\delta\Lambda}{\Lambda} c_3 L)(\omega^2 + \lambda(x)) \right)^{-1}) | x, t = 0 \rangle \tag{13}$$

such that the Green's function computed from the second self-consistency equation agrees with the Green's function computed from the first self-consistency equation averaged over disorder at large wavevector. Here we define $L = c_1^2 \Lambda^{8-2D} S$ where $c_1 = \frac{1}{\pi}^{D/2} \frac{\Gamma(D-2)}{\Gamma(2-D/2)\Gamma^2(D/2-1)}$, $c_2 = (1-4/d)c_3$, and $c_3 = 2\frac{\pi^{d/2}}{\Gamma(d/2)}\Lambda^{d-4}$.

The results above, to one loop, were obtained by considering the large wavevector fluctuations in λ due to the large wavevector fluctuations in $\delta\sigma^2$, and then finding how they renormalize the self-energy and vertex. To lowest order, one obtains the fluctuations in λ by inverting a polarization bubble. That is, one expands the self-consistency equation to linear order in λ to solve for large wavevector fluctuations in λ as a function of fluctuations in $\delta\sigma^2$. One finds then that

$$\delta\sigma^2(p) = c_1^{-1} p^{D-4}\lambda(p) + \dots \tag{14}$$

From this, we obtain fluctuations in λ at wavevector Λ which, to lowest order, are Gaussian with mean-square L. See figures 1,2, and 3. For more details, see previous work [4].

It may easily be seen that, to lowest order, the addition of the term $w(x)$ does not produce any additional large wavevector fluctuations in λ, as equation (14) is still true to lowest order in $w(x)$ and $\lambda(x)$. However, the term $w(x)$ can produce a renormalization of the self-energy. See figure 4. The result is to produce a term in the self-energy equal to

$$\Sigma(p, \omega) = -\delta\Lambda\omega^2 \int\limits_{k^2 = \Lambda^2} d^{d-1}k \frac{1}{(p+k)^2 + \omega^2} W \tag{15}$$

This is equal to

$$-\frac{\delta\Lambda}{\Lambda}\omega^2 W c_4 + \dots \tag{16}$$

where $c_4 = \Lambda^{d-2} 2\frac{\pi^{d/2}}{\Gamma(d/2)}$.

There is one other term that must be included in the RG flow equations at this order. The fluctuations in λ due to the fluctuations in $\delta\sigma^2$ can renormalize the vertex involving $w(x)$. See figure 5. This will change the term $w(x)\partial_t$ in the self-consistency equation to $w(x)\partial_t(1 + \frac{\delta\Lambda}{\Lambda}c_3 L)$. Note that the renormalization of the $w(x)$ term is equal to the renormalization of the ω^2 and $\lambda(x)$ terms in the self-consistency equation.

Putting all the terms together, we find that with a lowered cutoff $\Lambda - \delta\Lambda$, the renormalized theory is described by the new self-consistency equation

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$$(1 - \delta_3)\sigma^2(x) = \int d\omega \, \langle x, t = 0 | \Big(-(1 + \delta_2)\partial_x^2 + (1 + \delta_4)\omega^2 + i(1 + \delta_3)w(x)\omega + (1 + \delta_3)\lambda(x) \Big)^{-1})|x, t = 0\rangle \tag{17}$$

where $\delta_3 = \frac{\delta\Lambda}{\Lambda} c_3 L$, $\delta_2 = \frac{\delta\Lambda}{\Lambda} c_2 L$, and $\delta_4 = \delta_3 + \frac{\delta\Lambda}{\Lambda} W c_4$. Rescaling ω by $(1 + \frac{\delta_2 - \delta_4}{2})$ to make the coefficients in front of the ω^2 and ∂_x^2 terms the same, rescaling λ, and then rescaling the spatial scale to return the cutoff to Λ we find

$$\tilde{\sigma}^2(x) = \int d\omega \, \langle x, t = 0 | \Big(-\partial_x^2 + \omega^2 + i\tilde{w}(x)\omega + \lambda(x) \Big)^{-1})|x, t = 0\rangle \tag{18}$$

where

$$\tilde{\sigma}^2 = (1 - \delta_3 + \delta_2 + \frac{\delta_4 - \delta_2}{2} + (1 + \epsilon)\frac{\delta\Lambda}{\Lambda})\sigma^2(x) \tag{19}$$

$$\tilde{w}(x) = i(1 + \delta_3 - \delta_2 + \frac{\delta_2 - \delta_4}{2})w(x) \tag{20}$$

From this, we extract RG flow equations for σ_0^2, S, and W. The result is

$$\frac{d\ln\sigma_0^2}{d\ln\Lambda} = 1 + \epsilon - c_3 L + c_2 L + W\frac{c_4}{2} \tag{21}$$

$$\frac{d\ln S}{d\ln\Lambda} = \epsilon - 2c_3 L + 2c_2 L + Wc_4 \tag{22}$$

$$\frac{d\ln W}{d\ln\Lambda} = -\epsilon + 2c_3 L - 2c_2 L - Wc_4 \tag{23}$$

The renormalization group flow has a fixed line, as the product SW is invariant under the RG flow. It may be verified that the ratio S/W has a stable fixed point under RG flow for any ϵ and any value of SW. Further, it may be seen that the critical exponent ν on the fixed line is given by $\nu d = 2$, as if S is constant under RG flow, then σ_0^2 has $\frac{d\ln\sigma_0^2}{d\ln\Lambda} = 1 + \frac{\epsilon}{2}d/2$. Later, we will consider the effect of $1/N$ corrections.

First, note that the line is peculiar to having one time dimension. For fewer than one time dimension, there will be a stable fixed point at $W = 0$, which is attractive in the W direction. Thus, in the framework of a double-dimensional expansion, one may not see problems at low orders, as the fixed point has nice behavior for small numbers of time dimensions. Compare to results in the double-dimensional expansion [5].

Further, the presence of the fixed line only required that the renormalization of the $w(x)\partial_t$ vertex was equal to the renormalization of the vertex on the left-hand side of the self-consistency equation defining σ^2. This equality will persist to all orders in a loopwise expansion via a Ward identity. Thus, we expect that the fixed line is an exact property of the large N theory.

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Let us consider the effect of $1/N$ corrections on this line. To lowest order in $1/N$, for weak disorder, the $1/N$ corrections only modify the naive scaling dimensions in the RG flow. The scaling dimension of $\overline{\phi}\partial_t\phi$ is not changed under $1/N$ corrections. However, the scaling dimension of $\overline{\phi}\phi$ is changed by an amount $\eta = \frac{32}{3\pi^2}\frac{1}{2N}$. Thus, $1/N$ corrections will change the RG equations to

$$\frac{d\ln S}{d\ln\Lambda} = 2\eta + \epsilon - 2c_3L + 2c_2L + Wc_4 \tag{24}$$

$$\frac{d\ln W}{d\ln\Lambda} = -\epsilon + 2c_3L - 2c_2L - Wc_4 \tag{25}$$

Then, we find that SW is growing under the RG flow, and the system goes off to a different fixed point. The most reasonable guess then is that in a system with finite N (including physical systems with $N = 1$), the transition is not near the MI-SF transition, but is instead of another type, perhaps the phase-only transition. Other authors have shown that, in some cases, the phase-only transition is stable against weak commensuration effects [3].

IV. INSTANTON CALCULATIONS

Unfortunately, the ability to carry out instanton calculations in this system is rather limited. It will not be possible to calculate the action for the instanton with any precision, but we will at least present some arguments about the behavior of the instanton. The idea of the calculation is to look for configuration of $w(x)$ and $\sigma^2(x)$ (these configurations are the "instantons"), such that the self-consistency equation cannot be solved without including contributions from zero energy states, as discussed in section 2. Let us first consider the case in $2 + 1$ dimensions. Let us assume that we try to produce such states in a region of linear size L. Looking at the lowest energy state in this region, one would expect that the contribution of spatial gradient terms in the action would lead to an energy scale of order L^{-1}. The linear term $w(x)\omega$ in the action will become important, and produce such a zero energy state, when $w(x)\omega$ becomes of order ω^2. This implies occurs when $w(x)$ is of order ω, which implies $w(x) \approx L^{-1}$. For some appropriate configuration of $w(x)$, assuming quadratic fluctuations in $w(x)$ with strength of order W, we will have an action $S_{\text{instanton}} \propto \int \frac{w^2(x)}{W}d^2x$. Thus, these configurations will occur with exponentially small probability $e^{-S_{\text{instanton}}}$ for weak disorder in $w(x)$.

Away from $2 + 1$ dimensions, one will find that the action for the instanton, ignoring fluctuation corrections, is dependent on scale. For $d > 2$ it is increasing as the scale increases, indicating that large instantons are not present. For $d < 2$, it is decreasing as the scale increases, indicating that large instantons are easy to produce. This is simply a way of restating the fact that fluctuations in $w(x)$ are, at the pure fixed point, irrelevant for $d > 2$ and relevant for $d < 2$. It is to be expected that corrections due to fluctuations as considered in the renormalization group of the previous section will make the action for the instanton scale invariant. However, since we do not fully understand how to calculate instanton

corrections even in the simplest $d = 2$ case, the task of combining instanton and fluctuation corrections is presently hopeless.

V. HIGHER MOMENTS OF THE GREEN'S FUNCTION

Having considered the weakly commensurate case, and found no fixed point in physical systems, we return to the strongly commensurate case with $W = 0$, and consider the behavior of higher moments of the Green's function. A lowest order calculation will show log-normal fluctuations in the Green's function.

Let us first consider the second moment of the Green's function. That is, we would like to compute the disorder average of the square of the Green's function between two points, which we may write as $\langle G(0, x)^2 \rangle$. We may Fourier transform the square to obtain $\langle G^2(p, \omega) \rangle$. Now, one may, when averaging over disorder, include terms in which disorder averages connect the two separate Green's function. At lowest order, there is no low-momentum renormalization of the two Green's function propagator, beyond that due to the renormalization of each Green's function separately. See figure 6. That is, if one imagines the two Green's functions entering some diagram, with both Green's functions at low momentum, going through a sequence of scatterings, and exiting, again with both Green's functions at low momentum, one does not, to lowest order, find any contribution with lines connecting the two Green's function. The reason for this is that at this order we will only join the Green's functions with a single line, along which one must have momentum transfer of order Λ. This then requires that some of the ingoing or outgoing momenta must be of order Λ.

One does, however, find a contribution which we may call a renormalization of the vertex. See figure 7. In order to find the second moment of the Green's function, one must start both Green's functions at one point, and end both Green's functions at another point. Near the point at which both Green's functions start, one may connect both lines with a single scattering off of λ, at high wavevector of order Λ. Then, one can have a large momentum of order Λ circulating around the loop formed, while the two lines that leave to connect to the rest of the diagram still have low momentum. This then replaces the two Green's function vertex, which we will refer to as V_2, by a renormalized vertex.

The result of the above contribution is that the two Green's function vertex V_2 is renormalized under RG flow as

$$\frac{d\ln V_2}{d\ln\Lambda} = c_3 L = c_3 c_1^2 \Lambda^{8-2D} S \tag{26}$$

Then, the second moment of the Green's function, at momentum scale p is given in terms of the first moment by

$$\langle G^2(p, \omega) \rangle \propto \langle G(p, \omega) \rangle^2 p^{-2c_3 L} \tag{27}$$

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Note the factor of 2 in front of $c_3 L$, as the second moment of the Green's function gets renormalized at both vertices. One must insert the value of L at the fixed point into the above equation to obtain the behavior of the second moment.

For higher moments, the calculation is similar. In this case, one must, for the n-th moment, renormalize a vertex V_n. The result is

$$\frac{d\ln V_n}{d\ln\Lambda} = \frac{n(n-1)}{2}c_3 L = \frac{n(n-1)}{2}c_3 c_1^2 \Lambda^{8-2D} S \tag{28}$$

The factor $\frac{n(n-1)}{2}$ arises as at each stage of the RG one may connect any one of the lines in the vertex V_n to any other line in the vertex. There are $\frac{n(n-1)}{2}$ ways to do this. Then one finds

$$\langle G^n(p,\omega)\rangle \propto \langle G(p,\omega)\rangle^n p^{-n(n-1)c_3 L} \tag{29}$$

This result for the behavior of the higher moments of the Green's function is quite typical for disordered systems. Compare for example to the results on 2-dimensional Potts models [6]. From the results for the moments of the Green's function one may, under mild assumptions, determine the distribution function of the Green's function. This distribution function is the probability that, for a given realization of disorder, the Green's function between two points assumes a specific value. From the result for the moments given above one finds that the distribution function is log-normal. That is, the log of the function has Gaussian fluctuations. Physically this should be expected from any lowest order calculation, as lowest order calculations generally treat momentum scales hierarchically, and one is simply finding that at each scale there are random multiplicative corrections to the Green's function, causing the log of the Green's function to obey a random walk as length scale is increased.

VI. GLASSY BEHAVIOR IN THE LARGE N LIMIT

First, we would like to demonstrate that, in the large N limit, the self-consistency equation always has a unique solution. For generality, we consider here the case of quartic interactions instead of δ-function interactions. In the absence of terms linear in ω, uniqueness is clear on physical grounds, for the models considered above in which the coupling between neighboring fields ϕ is ferromagnetic and unfrustrated. However, we will show this to be true for any coupling between neighboring fields and in the presence of terms linear in ω.

Note that, for finite N, the terms linear in ω lead to frustration. Consider an $N = 1$ system with a finite number of sites. Assume that there is no hopping between sites, but there is some repulsion between sites due to a quartic term. Let there be terms linear in ω in the action, but no terms quadratic in ω. The states of the theory are then determined by how many particles occupy each site. The repulsion leads to an effective anti-ferromagnetic interaction, in the case in which each site has zero or one particles and we imagine one particle to represent spin up and no particles to represent spin down. This can then produce frustration. Compare to the Coulomb gap problem in localized electron systems [7].

However, physically speaking, as $N \rightarrow \infty$, the discreteness of particle number on each site disappears, and the system becomes unfrustrated. We will now show this precisely.

Consider a problem at non-zero temperature, so that there is a sum over frequencies ω. Consider an arbitrary single particle Hamiltonian H_0, defined on a V-site lattice, so that the self-consistency equation involves finding λ_i, where i ranges from 1 to V, such that

$$\sigma_i^2 + \sum_j M_{ij}\lambda_j = \sum_\omega \langle i|(H_0 + \lambda_i + iA_i\omega + B_i\omega^2)^{-1}|i\rangle \tag{30}$$

Here, σ_i^2 is a function of site i, and A_i and B_i are functions of site i defining the local value of the linear and quadratic terms in the frequency. The matrix M_{ij} is included to represent the effects of a quartic interaction. For the problem to be physically well defined, M_{ij} must be positive definite.

The proof that equation (30) has only one solution proceeds in two steps. First we note that if H_0 vanishes, then the equation obviously only has one solution with $\lambda \geq 0$. Next, we will show that as H_0 varies, λ_i varies smoothly, and therefore any arbitrary H_0 can be deformed smoothly into a vanishing H_0, leading to a unique solution for λ_i for arbitrary H_0.

Consider small changes δH_0 and $\delta \lambda_i$. In order for the self-consistency equation to remain true, if we define

$$v_i = -\sum_\omega \langle i|(H_0 + \lambda_i + iA_i\omega + B_i\omega^2)^{-1}\delta H_0(H_0 + \lambda_i + iA_i\omega + B_i\omega^2)^{-1}|i\rangle \tag{31}$$

we must have

$$v_i = \sum_\omega \langle i|(H_0 + \lambda_i + iA_i\omega + B_i\omega^2)^{-1}\delta\lambda_i(H_0 + \lambda_i + iA_i\omega + B_i\omega^2)^{-1}|i\rangle + \sum_j M_{ij}\lambda_j \tag{32}$$

The right hand side of equation (32) defines a linear function on $\delta\lambda_i$. If it can be shown that this function is invertible, then the theorem will follow. However, we have that

$$\text{Tr}(\delta\lambda_i \sum_\omega (H_0 + \lambda_i + iA_i\omega + B_i\omega^2)^{-1}\delta\lambda_i(H_0 + \lambda_i + iA_i\omega + B_i\omega^2)^{-1}) > 0 \tag{33}$$

due to the well known fact that second order perturbation theory always reduces the free energy of a quantum mechanical system with a Hermitian Hamiltonian at finite temperature. We also have, as discussed above, that M_{ij} is positive definite. Thus, the linear function on $\delta\lambda_i$ defined above is a sum of positive definite functions, and hence positive definite. Therefore, it is invertible and the desired result follows.

This result is interesting considering the phenomenon of replica symmetry breaking. It has been noticed by several authors that large N infinite-range spin glass models do not exhibit replica symmetry breaking within a meanfield approximation, both in the classical and quantum cases [8]. Although those calculations were based on the absence of unstable directions, in the large N limit, for fluctuations about the replica symmetric state, it is possible that the real reason for the absence of replica symmetry breaking is the uniqueness of the solution of the self-consistency equation, as shown above.

A second interesting question, having begun to consider possible glassy behavior in the large N limit, has to do with the nature of the ground state in the classical limit. If we drop all terms in ω, to produce a classical problem, and ask for the classical ground state, for some arbitrary bare Hamiltonian H_0, one may ask how many of the N available spin components will be used.

In this case, consider Hamiltonian H_0, which is a V-by-V matrix in the case where there are V sites. First consider the case in which H_0 is a real Hermitian matrix. Since we are considering arbitrary Hamiltonians H_0, we can, without loss of generality, constrain all spins to be the same length. We can find the classical ground state by looking for solutions of the self-consistency equation

$$\sigma^2 = \langle i|(H_0 + \lambda_i)^{-1}|i\rangle \tag{34}$$

in the limit as $\sigma^2 \to \infty$.

In this limit, the right-hand side will be dominated by zero energy states (more precisely, states that tend to zero energy as σ^2 tends to infinity) of the operator $H_0 + \lambda$. If the system has k of these states, the ground state of the system will use k of the spin components. If the system *needs* to use all k of these components to form a ground state, that is, ignoring the case in which a state using k spin components is degenerate with a state using fewer components, then even under small deformations of H_0 the system will use k spin components in the ground state. Then, under these small deformations, $H_0 + \lambda$ will still have k zero eigenvalues. To produce k zero eigenvalues for all real Hermitian matrices in a neighborhood of a given Hermitian matrix H_0 requires $k(k+1)/2$ free parameters. The elements of λ provide these parameters. Since there V of these elements, we find that $k(k+1)/2 \leq V$, and the number of spin components needed to form the classical ground state is at most $\sqrt{2V}$.

If H_0 were an arbitrary Hermitian matrix, with complex elements, or a symplectic matrix, one would find a similar result, with k still at most order $\sqrt{V}$, although the factor 2 would change. This is analogous to the different universality classes in random matrix theory [9]. Finally, we make one note on the number of parameters available to solve the self-consistency equation. There are V free parameters. However, self-consistency requires solving V independent equations, so the number of variables matches the number of equations. By considering the number of parameters requires to produce zero eigenvalues of $H_0 + \lambda$, we were able to obtain a bound on the number of zero eigenvalues. Still, one might wonder if there are enough free parameters to produce multiple zero eigenvalues and still solve the self-consistency equations, as it appears that one would then need $k(k+1)/2 + V$ free parameters. However, if there are k zero eigenvalues, by considering the different ways of populating the zero energy states (that is, considering the different ways in which the eigenvalues tend towards zero as σ^2 tends toward infinity) one obtains an additional $k(k+1)/2$ parameters, so the number of parameters available always matches the number of equations.

We can extend this theorem to look at metastable states. Suppose a configuration of spins is a local extremum of the energy H_0, for fixed length of spins. Then, since the derivative of the energy vanishes, one finds that a matrix $(H_0 + \lambda_i)$ must have a number of zero eigenvalues equal to the number of

spin components used. Suppose that for small deformations of H_0 there is still a nearby local minimum, as one would like to require for a stable state. Then, one can argue that the number of spin components k used in the state obeys $k(k+1)/2 \le V$.

This second theorem may be of interested in considering the onset of replica symmetry breaking. If we have a system in a large volume and large N limit, one must ask in which order the limits are taken. If the $N \to \infty$ limit is taken first, there will be no replica symmetry breaking. However, if the infinite volume limit is taken first, there may be replica symmetry breaking. If one has $N \ge 2k_{\max}$, where $k_{\max}$ is the largest k such that $k(k+1)/2 \le V$, then there are no local minima other than the ground state. This follows because, as shown above, a local extremum of the energy, ϕ_i, will use at most $k_{\max}$ spin components, and the ground state, ϕ_i^{gr}, can be constructed using a different set of $k_{\max}$ spin compoents. Then starting from ϕ_i, one finds that deforming the state along the path $\sqrt{1-\delta^2}\phi_i + \delta\phi_i^{\mathrm{gr}}$ as δ goes from 0 to 1 provides an unstable direction for fluctuations.

VII. CONCLUSION

In conclusion, we have considered the large N dirty boson model, including the effects of local incommensuration (the terms linear in ω). In the large N limit, a fixed line under RG is found, but is destabilized by including $1/N$ corrections. This suggests that the phase transition in experimental ($N = 1$) systems is of the phase-only type, instead of the MI-SF type.

There is a problem with local incommensuration in a perturbative approach, as discussed in the section on the Bose glass in the large N limit and the section on instanton calculations. One would like a quantitative method of assessing the results of the instantons, although this is largely a technical issue, as it appears that there are no accessible fixed points in the RG using this approach.

In the strongly commensurate case, it has been shown that one can calculate higher moments of the correlation functions. The result shows that the correlation functions have a log-normal distribution.

The two theorems proved in the last section give useful information on the relevance of the large N expansion in frustrated problems. It would be interesting to use these results as a starting point for a better understanding of replica symmetry breaking.

The large N approximation has been a useful approximation for pure, unfrustrated systems. It is hoped that it may become as useful for disordered interacting systems.

VIII. ACKNOWLEDGEMENTS

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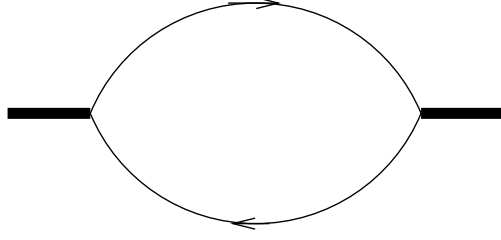
FIG. 1. Polarization bubble. Thick lines represent either scattering vertex off λ or scattering vertex used to define σ^2 in self-consistency equation.
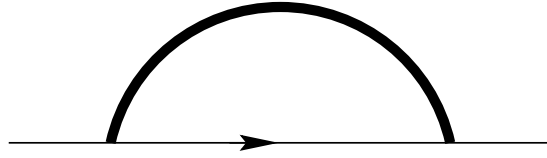

FIG. 2. Self-energy correction due to fluctuations in λ. Joining the thick lines in a loop denotes averaging λ over disorder in σ^2. Momentum of order Λ flows around loop.
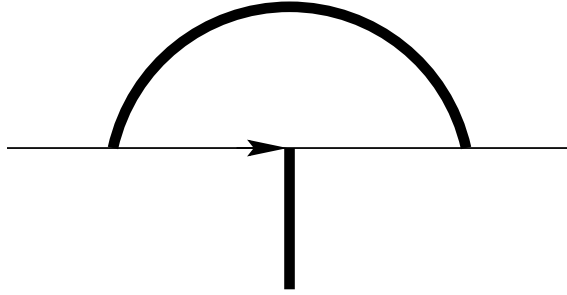

FIG. 3. Vertex correction due to fluctuations in λ. This represents both renormalization of vertex defining scattering off of λ and renormalization of vertex defining σ^2 in self-consistency equation.
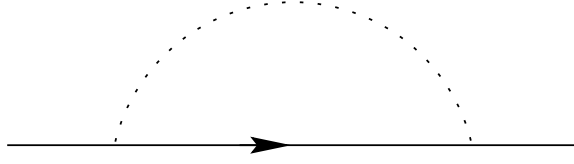
FIG. 4. Renormalization of ω^2 term due to fluctuations in $w(x)$. The vertices with thin lines represent scattering off of $w(x)\partial_t$, while the joining of the vertices represents averaging $w(x)$ over disorder.

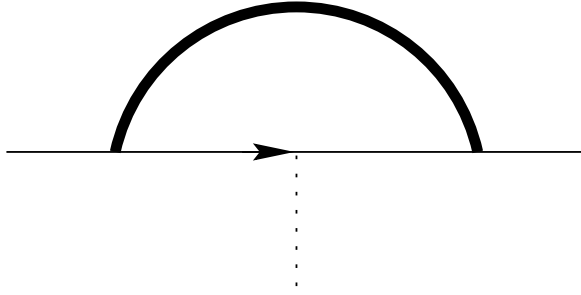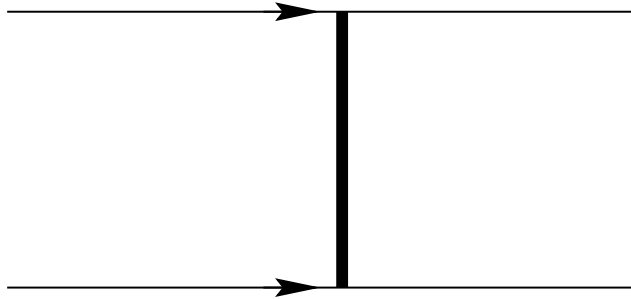FIG. 5. Renormalization of vertex for $w(x)$ due to fluctuations in λ.

FIG. 6. Possible contribution to propagation of two Green's functions, both in same realization of disorder. This is used to compute second moment of the Green's function. This diagram does *not* lead to any renormalization of low momentum behavior.
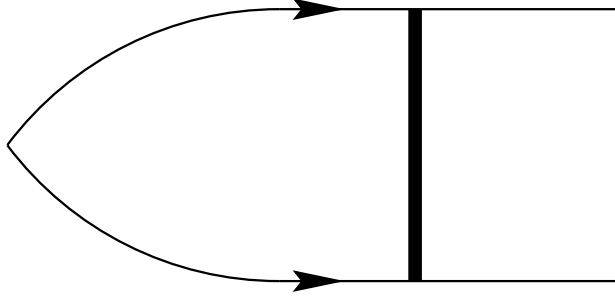
FIG. 7. Renormalization of vertex in computing higher moments of Green's function. Two Green's functions start at the same point. After Fourier transforming, this implies that they start with given total momentum. By including fluctuations in λ, with momentum of order Λ running around the loop, one can define a renormalized vertex.